Esercitazione 1

sabato 18 novembre 2023 10:11

(m, n) $\{(x,y) \mid |y| \le \frac{1}{x},\ x \ge 0\}$

$|y| \le 2$

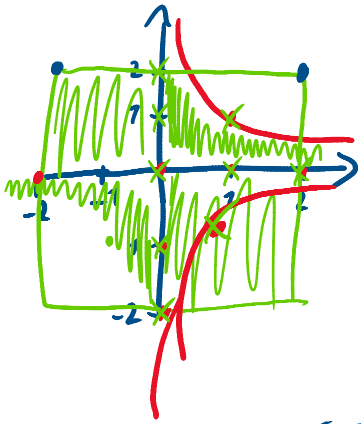
$|x| \le 2$

Sol

$-\frac{1}{x} \le y \le \frac{1}{x}$

$-2 \le y \le 2$

$-2 \le x \le 2$



(0,0)
(1,0)
(2,0)
(1,1)
(0,-2)
(0,-1)
(0,1)
(0,2)
(1,-1)

$$X = \begin{pmatrix} 0 & 1 & 2 \\ \frac{5}{9} & \frac{3}{9} & \frac{1}{9} \end{pmatrix} \qquad \frac{5+10+15+10+5}{81} + \frac{9+3}{81}$$

$$Y = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{9} & \frac{2}{9} & \frac{3}{9} & \frac{2}{9} & \frac{1}{9} \end{pmatrix} \Rightarrow E(Y) = 0$$

$$XY = \begin{pmatrix} -4 & -2 & -1 & 0 & 1 & 2 & 4 \\ \frac{1}{81} & \frac{5}{81} & \frac{6}{81} & \frac{5+}{81} & \frac{6}{81} & \frac{5}{81} & \frac{1}{81} \end{pmatrix}$$

SONO NON CORR

$E(Y)E(X) = E(XY)$

$$P(+2, 2) = 0 \neq \underbrace{P(+2)}_{X}\ \underbrace{P(2)}_{Y} = \frac{1}{81}$$

NON SONO INDIPEND

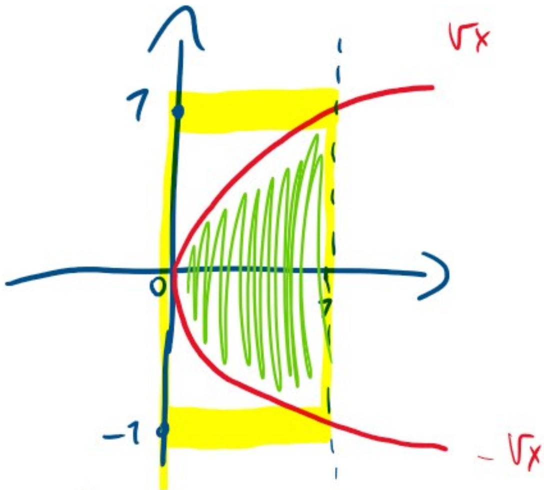
E2

$Z = (X, Y)$ unif distr on A

$$A = \{ (x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \ |y| \leq \sqrt{x} \}$$

Sol

$$-\sqrt{x} \leq y \leq \sqrt{x}$$



$$A = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy \, dx =$$

$$= \int_0^1 2\sqrt{x} \, dx = 2 \left. \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^1 = 2 \left(\frac{1}{\frac{3}{2}} - 0 \right) =$$

$$= 2 \cdot \frac{2}{3} \cdot 1 = \frac{4}{3}$$

$$A = \ldots = \frac{4}{3}$$

$$f(x,y) = \begin{cases} \frac{1}{A} = \frac{1}{\frac{4}{3}} = \frac{3}{4} & \text{on } A \\ 0 & \text{elsewhere} \end{cases}$$

$$f_1(x) = \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y) \, dy = \frac{3}{4} \int_{-\sqrt{x}}^{\sqrt{x}} 1 \cdot dy = \frac{3}{2} \sqrt{x}$$

$$|y| < |\sqrt{x} \Rightarrow x \geq |y|^2 \Rightarrow x \geq y^2$$

$|y| \leq \sqrt{x} \Rightarrow x \geq |y|^2 \Rightarrow x \geq y^2$

$$\Rightarrow f_2(y) = \int_0^{y^2} \frac{3}{4}\, dx = \frac{3}{4} y^2$$

Ons

$$f_1(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{2}\sqrt{x} & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

$$f_2(y) = \begin{cases} 0 & y < -1 \\ \frac{3}{4} y^2 & -1 \leq y \leq 1 \\ 0 & y > 1 \end{cases}$$

$$E(X) = \int_0^1 \frac{3}{2}\sqrt{x} \cdot x \, dx = \frac{3}{2} \int_0^1 x^{\frac{3}{2}} = \frac{3}{2} \left. \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right|_0^1 =$$

$$= \frac{3}{2} \cdot \frac{2}{5} \cdot 1 = \frac{3}{5}$$

$$E(Y) = \int_{-1}^1 \frac{3}{4} y^2 \cdot y \, dy = \frac{3}{4} \cdot \left. \frac{y^4}{4} \right|_{-1}^1 = 0$$

$$E(X) \cdot E(Y) = 0$$

$$E(XY) = \int_0^1 \int_{-1}^1 \frac{3}{4} xy = \frac{3}{4} \int_0^1 \int_{-1}^1 xy \, dy \, dx =$$

$$= \frac{3}{2} \int^1 x \cdot \left. \frac{y^2}{2} \right|^1 = 0$$

$$= \frac{3}{4} \int_0^1 x \cdot \frac{y^2}{2} \Big|_{-1}^{1} = 0$$

SONO <u>NON</u> CORRELATE

Ora

$R = [0,1] \times [-1,1]$

$\text{Area}(R) = 2$

Ma $\text{Area}(R) \neq \bar{A}$

Dunque

$f_1(x) f_2(y) > 0$ sul rettangolo R, allora

$\exists$ un insieme di misura positiva t.c. $f_1(x) \cdot f_2(y) \neq f(x,y)$

$\Rightarrow$ NON SONO INDIPENDENTI

$\blacksquare$

Esercitazione 2

domenica 19 novembre 2023 10:29

APPROSSIMAZIONE NORMALE

17)

Funzione Gaussiana

$p = p$

$q = 1 - p$

$\sigma^2 = p \cdot q$

$$P(X_1 + X_2 + \dots + X_n \le {}^*\text{Numero}^*) \approx \phi\Bigg(\underbrace{\frac{{}^*\text{Numero}^* \pm \overbrace{{}^*\text{Correzione Normale}^*}^{= \frac{1}{2}} - (n \cdot p)}{\sqrt{p \cdot q}\ \sqrt{n}}}_{=: K}\Bigg)$$

$$\Phi(K) \approx {}^*\text{Numero}_2{}^*$$

devo ricercare i valori della tavola

Se $K < 0 \Rightarrow$ Dobbiamo trovare

PASSIAMO AL MODULO

$$1 - \phi(|K|)$$

MASSIMA VEROSIMIGLIANZA

19)

$$f(k,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

Deriviamo rispetto a p

$$\frac{\partial}{\partial p} f(k,p) = \binom{n}{k}\left[k\, p^{k-1}(1-p)^{n-k} - p^k \cdot (n-k)(1-p)^{n-k-1}\right]$$

Imponiamo l'uguaglianza a zero e troviamo p

Chiameremo tale p come $\hat{p}$

20) Caso esponenziale

22) Caso esponenziale

$$L(x_1, \dots, x_n) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$\ell = \log(L)$$

derivo ℓ rispetto a λ

$$\frac{\partial \ell}{\partial \lambda}$$

Si impone uguale a zero e si trova λ

T STUDENT

23) MEDIA

$\bar{X}$ = *Numero* , n = *numero campioni*

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\bar{X} - X_i)^2 = \dots = \qquad S = \sqrt{S^2}$$

L'intervallo di fiducia per α è dato da:

$$\left[\bar{X} - \frac{S}{\sqrt{n}} t^{(n-1)}_{1-\frac{\alpha}{2}} , \bar{X} + \frac{S}{\sqrt{n}} t^{(n-1)}_{1-\frac{\alpha}{2}} \right]$$

$t^{(n-1)}_{1-\frac{\alpha}{2}}$ se l'intervallo è bilaterale

$t_{1-\alpha}(n-1)$ se l'intervallo è unilaterale

↗ ad esempio

$1-\alpha = 95\% \Rightarrow \alpha = 1-95\%$

Se bilaterale α la moltiplico per $\frac{1}{2}$

Se unilaterale mi tengo quell'α

$t_{1-\frac{\alpha}{2}}(n-1)$ = *INCROCIO $V=n-1$ con un "adeguato" α*

Sostituisco tutto nell'intervallo

TEST χ^2

26) m = suddivisione dell'insieme (o tuple)

$\bar{P}_i$ = Probabilità dell'i-esimo campione

$$T_n = n \cdot \sum_{i=1}^{m} \frac{(\bar{P}_i - P_i)^2}{P_i}$$

n: campione

P_i = Probabilità di ogni suddivisione

$T_n = \dots =$ *Numero*

Calcoliamo ora $\chi^2_{0,95}(m-1)$

Calcoliamo ... $\chi^2_{0,95}$ ...

per esempio

- Se $T_n \leq \chi^2_{\text{"alfa"}}(n-1) \Rightarrow$ NON POSSO SMENTIRE L'IPOTESI
- Se $T_n > \chi^2_{\text{"alfa"}}(n-1) \Rightarrow$ POSSO SMENTIRE

$\blacksquare$

χ^2 DELL' INDIPENDENZA

28)

	n_1	n_2	...	n_n		
m_1	x_1	x_2	...	x_n	$x_1+x_2+\dots x_n$	a'
m_2	y_1	y_2	...	y_n	$y_1+y_2+\dots y_n$	b'
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
m_n	z_1	z_2	...	z_n	$z_1+z_2+\dots z_n$	z'
	$x_1+y_1+\dots+z_1$	$x_2+y_2+\dots+z_2$	...	$x_n+y_n+\dots+z_n$	TOT	$(a+b+\dots+z)$ $(a'+b'+z')$
	a	b	...	z		

Devo calcolare le tabelle delle freq congiunte π_{ij}:

n	n_1	n_2	..	j

	n_1	n_2	$\dots$	$\downarrow$	
n_1	$\frac{X_1}{TOT}$	$\frac{X_2}{TOT}$	$\vdots$	$\dots$	$\frac{X_1}{TOT} + \frac{X_2}{TOT} + \dots + \frac{X_n}{TOT}$ $\quad J_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	J_2
n_n	$\frac{Z_1}{TOT}$	$\frac{Z_2}{TOT}$	$\vdots$	$\dots$	$\vdots$
$i \rightarrow$	$\frac{X_1}{TOT} + \frac{Y_1}{TOT} + \dots + \frac{Z_1}{TOT}$	$\dots$			J_n
	i_1	i_2	$\dots$	i_n	

Devo calcolare la tabella dei prodotti marginali $\bar{p}_i \, \bar{q}_j$:

	n_1	n_2	
n_1	$J_1 \, i_1$	$J_1 \, i_2$	$\dots$
n_2	$J_2 \, i_1$	$J_2 \, i_2$	$\dots$

$$T_n = n \sum_{i,j} \frac{\left((J_i \, i_j) - \pi_{ij} \right)^2}{\pi_{ij}}$$

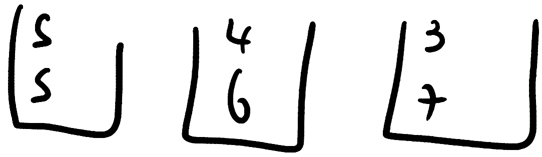
Troviamo χ^2:

$$\chi^2_{0,95}\left((\text{numero righe} - 1)(\text{numero colonne} - 1) \right)$$

Appello 7/12/21

martedì 21 novembre 2023 15:39

Es 2



$$P(U_1|B) = \frac{P(B|U_1)\,\cancel{P(U_1)}}{\sum_{i=1}^{3} P(B|U_i)\,\cancel{P(U_i)}} =$$

$$= \frac{\frac{5}{10}}{\frac{5}{10}+\frac{4}{10}+\frac{3}{10}} = \frac{5}{12}$$

$$P(U_2|B) = \dots = \frac{4}{12}$$

$$P(U_3|B) = \dots = \frac{3}{12}$$

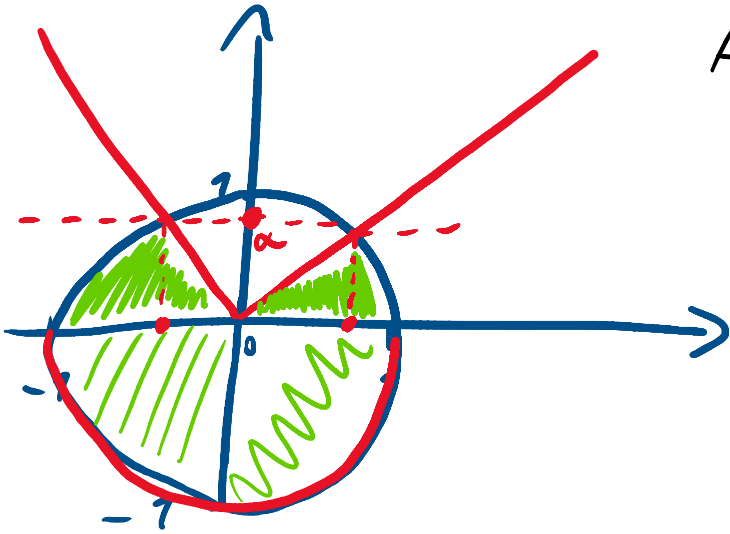
$$P(BB) = P(B|U_1')\cdot P(U_1') + P(B|U_2')\,P(U_2') + P(B|U_3')\,P(U_3')$$

$$= \frac{4}{9}\cdot\frac{5}{12} + \frac{3}{9}\cdot\frac{4}{12} + \frac{2}{9}\cdot\frac{3}{12} = \frac{19}{54}$$

$$-\sqrt{1-y^2} < x < \sqrt{1-y^2}$$

Es 3

$$A = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \ , \ y \leq |x| \}$$



Area Cerchio $= \pi r^2$

$= \pi$

$A = \frac{3}{4}\pi$

$$f(x,y) = \begin{cases} \frac{4}{3\pi} & \text{in } A \\ 0 & \text{altrove} \end{cases}$$

$$f_1(x) = \int_{-\sqrt{1-x^2}}^{|x|} f(x,y)\, dy = \frac{4}{3\pi} \cdot \left(|x| + \sqrt{1-x^2} \right)$$

$$f_2(y) = \int f(x,y)\, dx =$$

Es 4

$$P(X_1 + X_2 + \dots + X_{100} \geq 40)$$

$\Rightarrow$

$$1 - P(X_1 + X_2 + \ldots + X_{100} < 40)$$

$$\Rightarrow 1 - P(X < 40) \cong 1 - \phi\left(\frac{39{,}5 - 50}{5}\right) =$$

$$\cong 1 - \phi\left(-\frac{10{,}5}{5}\right)$$

$$\cong 1 - \left(1 - \phi\left(\frac{10{,}5}{5}\right)\right) = \phi\left(\frac{10{,}5}{5}\right) = \phi(2{,}1) =$$

$$= 0{,}98214$$

$\blacksquare$

Appello 19/7/19

martedì 21 novembre 2023 16:24

Es 1

x^3

y^3

z^3

$x^2 \begin{cases} y \\ z \end{cases}$

$y^2 \begin{cases} x \\ z \end{cases}$

$z^2 \begin{cases} x \\ y \end{cases}$ TOT = 20

$x \begin{cases} y \\ z \end{cases}$

$y \begin{cases} x \\ z \end{cases}$

z

xyz

Es 2

D6 e D4

4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

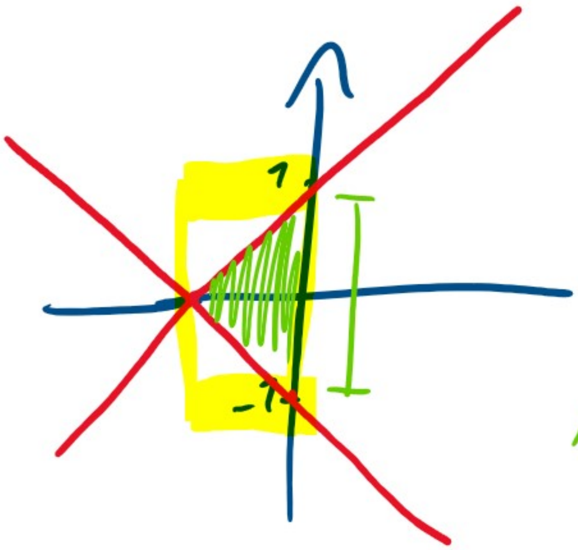
$\frac{4}{24} = \frac{1}{6}$

$\frac{6+1}{2} + \frac{4+1}{2}$

$3,5 + 2,5 = 6$

Es 3

$$A = \{(x,y) \in \mathbb{R}^2 \mid |y| \leq x+1 \quad , \; x \leq 0\}$$



$-x-1 \leq y \leq x+1$

$|y|-1 \leq x \leq 0$

$$A = \frac{2 \cdot 1}{2} = 1$$

$$f(x,y) = \begin{cases} 1 & \text{su } A \\ 0 & \text{altrove} \end{cases}$$

$$f_1(x) = \int_{-x-1}^{x+1} f(x,y)\, dy = x+1-(-x-1) = 2x+2$$

$$f_1(x) = \begin{cases} 2x+2 & -1 \leq x \leq 0 \\ 0 & \text{altrove} \end{cases}$$

$$f_2(y) = \int_{|y|-1}^{0} f(x,y)\,dx = -|y|+1 = 1-|y|$$

$$f_2(y) = \begin{cases} 1-|y| & -1 \le y \le 1 \\ 0 & \text{altrove} \end{cases}$$

$$E(X) = \int_{-1}^{0} (2x+2)x\,dx = 2\frac{x^3}{3}\Big|_{-1}^{0} + x^2\Big|_{-1}^{0} =$$

$$\frac{2}{3}(x^3)\Big|_{-1}^{0} + x^2\Big|_{-1}^{0} =$$

$$\frac{2}{3}\left(0-(-1)\right) + \left(0-(-1)^2\right) =$$

$$\frac{2}{3}\cdot 1 + (-1) = \frac{2}{3} - 1 = -\frac{1}{3}$$

$$E(Y) = \int_{-1}^{1} 1-|y|\,dy = \int_{-1}^{0} 1+y\,dy + \int_{0}^{1} 1-y\,dy =$$

$$= y + \frac{y^2}{2}\Big|_{-1}^{0} + y - \frac{y^2}{2}\Big|_{0}^{1} =$$

$$= \frac{1}{2} + \left(1 - \frac{1}{2}\right) = 1$$

$$E(XY) = \int_{-1}^{0}\int_{-1}^{1} f(x,y)\, xy \, dx\, dy = \int_{-1}^{0}\int_{-1}^{1} xy \, dy\, dx =$$

$$= \int_{-1}^{0} x \, \frac{y^2}{2}\Big|_{-1}^{1} dx = \int_{-1}^{0} x \cdot 0 = 0$$

$E(X)\cdot E(Y) \neq E(XY)$ $\Rightarrow$ NON CORREL

$f_1(x)\, f_2(y) > 0$ sul rettangolo $R = [-1,0] \times [-1,1]$ che ha misura 2, mentre il supporto della densità congiunta ha misura $A = 1$

$\exists$ quindi un insieme di misura positiva dove $f(x\,y) \neq f_1(x)\, f_2(y)$

$\blacksquare$

Es 4

$n = 100$

$[0, \frac{1}{4}]$	$(\frac{1}{4}, \frac{1}{2}]$	$(\frac{1}{2}, \frac{3}{4}]$	$(\frac{3}{4}, 1]$
30	25	25	20

$$T_n = n \sum_{i=1}^{m} \frac{(\bar{p}_i - p_i)^2}{p_i}$$

$$\Rightarrow$$

$$T_{100} = 100 \left(\frac{\left(\frac{30}{100} - \frac{25}{100}\right)^2}{\frac{25}{100}} + \cancel{2\frac{\left(\frac{25}{100} - \frac{25}{100}\right)^2}{\frac{25}{100}}} + \frac{\left(\frac{20}{100} - \frac{25}{100}\right)^2}{\frac{25}{100}} \right)$$

$$= 100 \cdot \left(\frac{\left(\frac{5}{100}\right)^2}{\frac{25}{100}} + \frac{\left(-\frac{5}{100}\right)^2}{\frac{25}{100}} \right) = \dots = 2$$

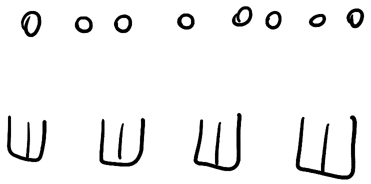
$$\chi^2_{0,95}(3) = 7,815$$

$\Rightarrow$ NON POSSO SMENTIRE

Appello 19/7/2019

martedì 21 novembre 2023 17:05

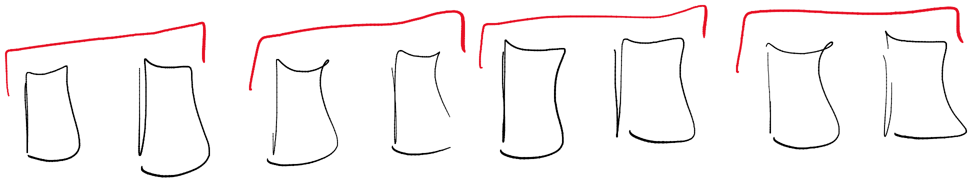
Es 1



$$\binom{8}{2}+\binom{6}{2}+\binom{4}{2}+\binom{2}{2}$$

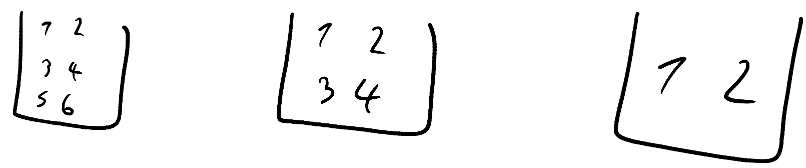
$$= 28 + 15 + 6 + 1 = 50$$

$$\frac{8!}{4!(2!)^4} = \frac{\cancel{8}\cdot 7\cdot \cancel{6}^{3}\cdot 5\cdot \cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{16}\ \cancel{2}} = 105$$



$$\frac{8!}{4!\,(2)^4}$$

Es 2



$$P(U_1|A) = \frac{P(A|U_1)\,\cancel{P(U_1)}}{\sum_{i=1}^{3} P(A|U_i)\,\cancel{P(U_i)}} =$$

$$\sum_{i=1}^{3} P(A|U_i)\, \cancel{P(U_i)}$$

$$= \frac{\frac{4}{6}}{\frac{4}{6}+\frac{2}{4}+0} = \frac{4}{7}$$

$$\frac{1}{3}\Big(P(A|U_1) + P(A|U_2) + P(A|U_3) \Big) =$$

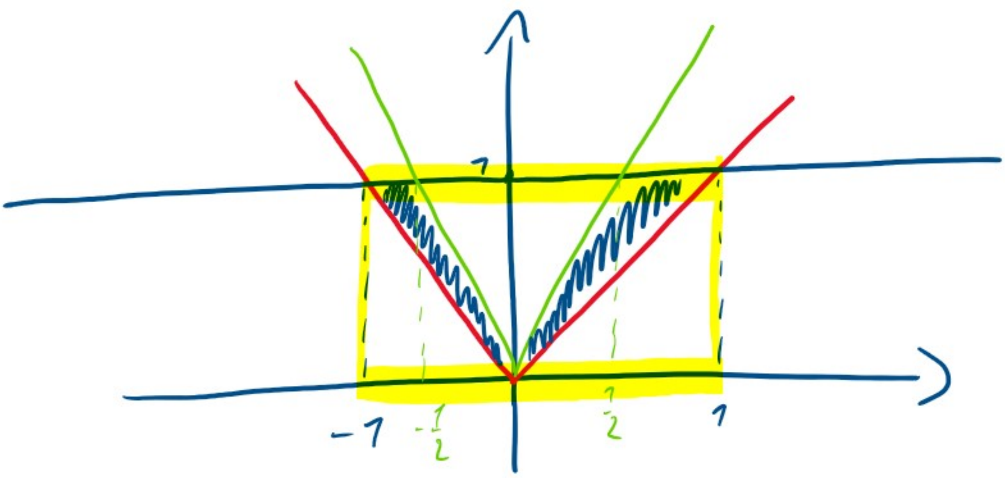
$$\frac{1}{3}\left(\frac{4}{6} + \frac{2}{4} + 0 \right) = \ldots$$ *Nenner*

Ex 3

$A = B \setminus C$

$B = \{ (x,y) \mid |x| \leq y \leq 1 \}$ $\quad -y \leq x \leq y$

$C = \{ (x,y) \mid 2|x| \leq y \leq 1 \}$ $\quad -\frac{y}{2} \leq x \leq \frac{y}{2}$



$$A = \frac{1}{2}$$

$$f(x,y) = \begin{cases} 2 & \text{on } A \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(x) = \int_{|x|}^{2|x|} f(x,y)\, dy = 2 \int_{|x|}^{2|x|} dy =$$

$$= 2 \left(y \right) \Big|_{|x|}^{2|x|} =$$

$$= 2 \left(2|x| - |x| \right) = 2|x|$$

$$\Rightarrow f_1(x) = \begin{cases} 2|x| & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = 2\int_{\frac{y}{2}}^{y} f(x,y)\, dx = 4\left(y - \frac{y}{2} \right) = 2y$$

$$f_2(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-1}^{1} 2|x| \cdot x\, dx = \int_{-1}^{0} -2x^2\, dx + \int_{0}^{1} 2x^2\, dx =$$

$$= -2\frac{x^3}{3}\Big|_{-1}^{0} + 2\frac{x^3}{3}\Big|_{0}^{1} =$$

$$= -2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) = 0$$

$$E(XY) = \int_{-1}^{1}\int_{0}^{1} f(x,y)\, xy \, dx\, dy =$$

$$= 2\int_{-1}^{1}\int_{0}^{1} xy \, dy\, dx =$$

$$= 2\int_{-1}^{1} x \, \frac{y^2}{2}\Big|_0^1 dx = 2\int_{-1}^{1} \frac{x}{2}\, dx = \int_{-1}^{1} x\, dx =$$

$$= \frac{x^2}{2}\Big|_{-1}^{1} = 0$$

Poiché

$$E(X)\, E(Y) = E(XY)$$

$\Rightarrow$ SONO CORRELATE

Ora $f_1(x)$ e $f_2(y)$ sono dipendenti in quanto

$f_1(x) \cdot f_2(y) > 0$ sul rettangolo $R = [-1,1] \times [0,1]$

che ha misura $A' = 2$

mentre il supporto della densità congiunta ha misura $\frac{1}{2}$

$\Rightarrow$

$\exists$ dunque un insieme di misura positiva dove

$$f_1(x) \cdot f_2(y) \neq f(x,y)$$

Es 4

	MASCHI	FEMMINE	
≥ 70	90	138	228
< 70	10	12	22
	100	150	250

Tabella delle frequenze congiunte $\pi_{i,j}$:

	MASCHI	FEMM	
≥ 70	0,36	0,552	0,972
< 70	0,04	0,048	0,088
	0,76	0,6	

Tabella Prodotti Marginali $\bar{p}_i \bar{q}_j$:

	MASCHI	FEMMINE
≥ 70	0,69312	0,5472
< 70	0,06688	0,0528

$$T_{250} = 250 \cdot \left(\frac{(0{,}69312 - 0{,}36)^2}{0{,}36} + \frac{(0{,}5472 - 0{,}552)^2}{0{,}552} + \frac{(0{,}06688 - 0{,}04)^2}{0{,}04} + \frac{(0{,}0528 - 0{,}048)^2}{0{,}048} \right) =$$

$$\chi^2_{0{,}95}(1) = 3{,}841$$